n阶行列式与n维向量的向量积、混合积

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内容提要

向量积定义

定义1.1 设$\epsilon_1, \epsilon_2, \cdots, \epsilon_n$是$n$维欧氏空间$V$的标准正交基,

$\alpha_j = \sum_{i=1}^{n} a_{ij}\varepsilon_i$, $j = 1, 2, \cdots, n-1$. 令

$$A_j = (-1)^{j+1} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1\ n-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j-1\ 1} & a_{j-1\ 2} & \cdots & a_{j-1\ n-1} \\ a_{j+1\ 1} & a_{j+1\ 2} & \cdots & a_{j+1\ n-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n\ 1} & a_{n\ 2} & \cdots & a_{n\ n-1} \end{vmatrix},$$

$j = 1, 2, \cdots, n.$

向量积定义

$n-1$个向量$\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$的向量积定义为

$$\begin{vmatrix} \epsilon_1 & a_{11} & a_{12} & \cdots & a_{1\ n-1} \\ \epsilon_2 & a_{21} & a_{22} & \cdots & a_{2\ n-1} \\ \vdots & \vdots & \vdots & \vdots & \\ \epsilon_n & a_{n\ 1} & a_{n\ 2} & \cdots & a_{n\ n-1} \end{vmatrix} = A_1\epsilon_1 + A_2\epsilon_2 + \cdots + A_n\epsilon_n,$$

记作$VP_\epsilon(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$, 其中$\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)^T$.

显然, $VP_\epsilon(\epsilon_1, \epsilon_2, \cdots, \epsilon_{n-1}) = (-1)^{n+1}\epsilon_n$.

向量积性质

命题1.2 设$\epsilon_1, \epsilon_2, \cdots, \epsilon_n$和$\eta_1, \eta_2, \cdots, \eta_n$都是$n$维欧氏空间$V$的标准正交基,

$\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$是$V$的一个向量组,

则$VP_\epsilon(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) = \pm VP_\eta(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$.

证明: 设$(\eta_1, \eta_2, \cdots, \eta_n) = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)G$, 则$\det(G) = \pm 1$.

设$\alpha_j = \sum_{i=1}^n a_{ij}\varepsilon_i = \sum_{i=1}^n b_{ij}\eta_i, j = 1, 2, \cdots, n-1$, 令

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1\ n-1} \\ a_{21} & a_{22} & \cdots & a_{2\ n-1} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n\ n-1} \end{pmatrix}, \ B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1\ n-1} \\ b_{21} & b_{22} & \cdots & b_{2\ n-1} \\ \vdots & \vdots & \ldots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{n\ n-1} \end{pmatrix},$$

则$A = GB$. 故

$$\begin{aligned} VP_\epsilon(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) &= \det(\epsilon, A) = \det(G^T\eta, G^TB) \\ &= \det(G^T)\det(\eta, B) = \pm VP_\eta(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}). \end{aligned}$$

向量积性质

命题1.2说明向量积与标准正交基的选择最多差一个负号. 因此, 下面都取定n维欧氏空间V的一个标准正交基为$\epsilon_1, \epsilon_2, \cdots, \epsilon_n$, 并将$n-1$个向量$\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$的向量积记为$VP(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$.

由行列式的性质, 我们有

命题1.3 若向量组$\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$线性相关, 则$VP(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) = 0$.

命题1.4 对于向量组$\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$, 若$\alpha_j = \sum_{i=1}^{m} k_i\beta_i$, $1 \le j \le n-1$, 则

$$VP(\alpha_1, \cdots, \alpha_j, \cdots, \alpha_{n-1}) = \sum_{i=1}^{m} k_i VP(\alpha_1, \cdots, \beta_i, \cdots, \alpha_{n-1}).$$

向量积性质

命题1.5 设S_{n-1}为$n-1$元对称群, 则对于任意$\sigma \in S_{n-1}$, 有

$$VP(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \cdots, \alpha_{\sigma(n-1)}) = \begin{cases} -VP(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}), & \text{若}\sigma\text{奇置换}, \\ VP(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}), & \text{若}\sigma\text{偶置换}. \end{cases}$$

命题1.6 设W是由向量组$\alpha_1, \alpha_2, \cdots, \alpha_m$生成的子空间, $m < n-1$, $P_W(\alpha_j)$表示α_j在W上的正交投影, $P_{W^\perp}(\alpha_j)$表示α_j在W的正交补$W^\perp$上的正交投影, $j > m$, 则

$$VP(\alpha_1, \cdots, \alpha_m, \alpha_{m+1}, \cdots, \alpha_{n-1}) = VP(\alpha_1, \cdots, \alpha_m, P_{W^\perp}(\alpha_{m+1}), \cdots, P_{W^\perp}(\alpha_{n-1})).$$

向量积性质

设$\alpha_1, \alpha_2, \cdots, \alpha_m$是$n$维欧氏空间的一个线性无关组, 经过标准正交化过程, 可以得到m阶上三角矩阵

$$T = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1m} \\ 0 & t_{22} & \cdots & t_{2m} \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \cdots & t_{mm} \end{pmatrix}$$

和规范正交向量组$\epsilon_1, \epsilon_2, \cdots, \epsilon_m$使得$(\alpha_1, \alpha_2, \cdots, \alpha_m) = (\epsilon_1, \epsilon_2, \cdots, \epsilon_m)T$, 其中$t_{ii} > 0,\ i = 1, 2, \cdots, m$.

命题1.7 $VP(\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_m) = \left[(-1)^{i+1} \prod_{j=1, j\neq i}^{m} t_{jj}\right] \epsilon_i,\ i = 1, 2, \cdots, m.$

混合积的定义

定义2.1　设$\epsilon_1, \epsilon_2, \cdots, \epsilon_n$是$n$维欧氏空间$V$的标准正交基, V的内积记为$\langle\cdot,\cdot\rangle$.

$\alpha_j = \sum_{i=1}^{n} a_{ij}\varepsilon_i$, $j = 1, 2, \cdots, n-1$. n个向量$\alpha_1, \alpha_2, \cdots, \alpha_n$的混合积定义为

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}A_{11} + a_{21}A_{21} + \cdots + a_{n1}A_{n1} = \langle \alpha_1, VP(\alpha_2, \alpha_3, \cdots, \alpha_n) \rangle,$$

记作$MP_\epsilon(\alpha_1, \alpha_2, \cdots, \alpha_n)$. 其中$\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)^T$.

显然, $MP_\epsilon(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) = 1$.

混合积的性质

命题2.2 设$\epsilon_1, \epsilon_2, \cdots, \epsilon_n$和$\eta_1, \eta_2, \cdots, \eta_n$都是$n$维欧氏空间$V$的标准正交基, $\alpha_1, \alpha_2, \cdots, \alpha_n$是$V$的一个向量组, 则$MP_\epsilon(\alpha_1, \alpha_2, \cdots, \alpha_n) = \pm MP_\eta(\alpha_1, \alpha_2, \cdots, \alpha_n)$.

命题2.2说明混合积与标准正交基的选择最多差一个负号. 因此, 下面都取定n维欧氏空间V的一个标准正交基为$\epsilon_1, \epsilon_2, \cdots, \epsilon_n$, 并将$n$个向量$\alpha_1, \alpha_2, \cdots, \alpha_n$的混合积记为$MP(\alpha_1, \alpha_2, \cdots, \alpha_n)$.

由行列式的性质, 我们有

命题2.3 $MP(\alpha_1, \alpha_2, \cdots, \alpha_n) = \langle \alpha_i, VP(\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_n) \rangle$, $i = 1, 2, \cdots, n$.

命题2.4 若向量组$\alpha_1, \alpha_2, \cdots, \alpha_n$线性相关, 则$MP(\alpha_1, \alpha_2, \cdots, \alpha_n) = 0$.

混合积的性质

命题2.5 对于向量组$\alpha_1, \alpha_2, \cdots, \alpha_n$, 若$\alpha_j = \sum_{i=1}^{m} k_i\beta_i$, $1 \le j \le n$, 则

$$MP(\alpha_1, \cdots, \alpha_j, \cdots, \alpha_n) = \sum_{i=1}^{m} k_i MP(\alpha_1, \cdots, \beta_i, \cdots, \alpha_n).$$

命题2.6 设S_n为n元对称群, 则对于任意$\sigma \in S_n$, 有

$$MP(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \cdots, \alpha_{\sigma(n)}) = \begin{cases} -MP(\alpha_1, \alpha_2, \cdots, \alpha_n), & \text{若}\sigma\text{奇置换}, \\ MP(\alpha_1, \alpha_2, \cdots, \alpha_n), & \text{若}\sigma\text{偶置换}. \end{cases}$$

混合积的性质

设$\alpha_1, \alpha_2, \cdots, \alpha_m$是$n$维欧氏空间的一个线性无关组, 经过标准正交化过程, 可以得到m阶上三角矩阵

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和规范正交向量组$\epsilon_1, \epsilon_2, \cdots, \epsilon_m$使得$(\alpha_1, \alpha_2, \cdots, \alpha_m) = (\epsilon_1, \epsilon_2, \cdots, \epsilon_m)T$, 其中$t_{ii} > 0,\ i = 1, 2, \cdots, m$.

命题2.7 $MP(\alpha_1, \alpha_2, \cdots, \alpha_m) = \pm \prod_{j=1}^{m} t_{jj}$.

结束

谢谢!